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New quantum squeezed states for the time-dependent harmonic oscillator

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Abstract
New generalized squeezed states for the time-dependent harmonic oscillator are found through the theory of invariants. Our method gives a comparatively clearer picture than methods using evolution operators because we can establish a direct connection between the classical and quantum solutions. The additional significance of our method is that it is possible to find new generalized quantum squeezed states from just one particular solution to the classical time-dependent oscillator. Accordingly, more general results for the variance of x (squared uncertainty) are found for the cases of linear sweep of the restoring force and compared to recent results encountered in the literature.

Keywords: Time-dependent harmonic oscillator, quantum squeezed states

The time-dependent harmonic oscillator (TDHO) continues to have widespread applications in various branches of physics [1–7]. Among numerous approaches to finding exact solutions contained by the TDHO, we point out the evolution operator method used recently by a number of authors [1–3]. Furthermore, the recent generation of coherent states of quantum fluids [5] and nonclassical squeezed states of the electromagnetic field [6] typify the continued importance of the TDHO. It has also been shown that displaced and squeezed number states of a simple harmonic oscillator can be generated by displacing the oscillator and changing its frequency [3]. In addition, it is worth noticing an earlier prescription for obtaining adiabatic invariants and coherent states for the TDHO using the Born–Fock adiabatic theorem for quantum operators to all orders [7].

Motivated by the earlier considerable interest in the TDHO, we explore in this work the possibility of new generalized squeezed states for the TDHO through the theory of exact invariants. Our method gives a comparatively clearer picture than methods using operators because we can establish a direct connection between the classical and quantum cases. The additional significance of our method is that it is possible to find new generalized quantum squeezed states from just one particular solution to the classical time-dependent oscillator. Accordingly, we find more general results for the variance of x (squared uncertainty) for the cases of linear sweep of the restoring force and compare them to recent results encountered in the literature [1–4].

To this end, a general and exact solution to the Schrödinger equation for the TDHO of frequency $\Omega(t)$:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + \left(\frac{1}{2}m\Omega^2(t)x^2\right)\psi(x,t) \quad (1)$$

can be found by first expressing the complex wavefunction in a polar form:

$$\psi(x,t) = \phi(x,t)\exp[(i/\hbar)S(x,t)] \quad (2)$$

where $\phi(x,t)$ and $S(x,t)$ are real functions. In doing so, equation (1) can be recast as

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{1}{2}m\Omega^2(t)x^2\right) - \frac{\hbar^2}{2m\phi}\frac{\partial^2\phi}{\partial x^2} = 0 \quad (3)$$

and

$$\frac{\partial\phi^2}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\phi^2}{m}\frac{\partial S}{\partial x}\right) = 0. \quad (4)$$

Since we are interested in the most general Gaussian wave packet solution to problem, we make the ansatz [8]

$$\phi(x,t) = \left(2\pi a^2(t)\right)^{-1/4}\exp\left[-\frac{[x-X(t)]^2}{4a^2(t)}\right] \quad (5)$$

where $a(t)$ and $X(t)$ are auxiliary functions of time, to be determined in what follows.

First, we substitute equation (5) into (4) and integrate the result to obtain

$$\frac{\partial S(x,t)}{\partial x} = \frac{m\dot{a}(t)}{a(t)}[x - X(t)] + m\dot{X}(t) \tag{6}$$

where the constant of integration must be zero since ϕ^2 and $[\phi^2(\partial S/\partial x)]$ vanish for $|x| \to \infty$. In fact, any well-behaved function (such as any polynomial function of $(x - X)$) multiplied by ϕ^2 clearly vanishes as $|x| \to \infty$. From equation (5) we can derive the useful expression

$$-\frac{\hbar^2}{2m\phi}\frac{\partial^2\phi}{\partial x^2} = -\frac{\hbar^2}{8ma^4(t)}[x - X(t)]^2 + \frac{\hbar^2}{4ma^2(t)}. \tag{7}$$

Next, the phase of the wavefunction can also be found by integrating equation (3) in time with the help of equations (6) and (7):

$$S(x,t) = \frac{m\dot{a}(t)}{2a(t)}[x - X(t)]^2 + m\dot{X}(t)[x - X(t)] + \int_0^t \mathrm{d}t' \left(\frac{1}{2}m\dot{X}^2(t') - \frac{1}{2}m\Omega^2(t')X^2(t') - \frac{\hbar^2}{4ma^2(t')}\right). \tag{8}$$

From equation (8) we can construct the following relations:

$$\frac{\partial S}{\partial t} = \frac{m}{2}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)(x - X)^2 + \left(-\frac{m\dot{a}\dot{X}}{a} + m\ddot{X}\right)(x - X) - m\dot{X}^2 + \frac{1}{2}m\dot{X}^2 - \frac{1}{2}m\Omega^2(t)X^2 - \frac{\hbar^2}{4ma^2} \tag{9}$$

and

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 = \frac{m\dot{a}^2}{2a^2}(x-X)^2 + \frac{m\dot{a}\dot{X}}{a}(x-X) + \frac{1}{2}m\dot{X}^2. \tag{10}$$

Finally, equation (4) can be recast with the help of equations (7), (13) and (10):

$$\frac{m}{2a}\left[\ddot{a} + \Omega^2(t)a - \frac{\hbar^2}{4m^2a^3}\right](x - X)^2 + m\left[\ddot{X} + \Omega^2(t)X\right](x - X) = 0. \tag{11}$$

This equation is satisfied once the coefficients of $(x - X)^2$ and $(x - X)$ are set equal to zero, namely

$$\ddot{a} + \Omega^2(t)a = \frac{\hbar^2}{4m^2a^3} \tag{12}$$

$$\ddot{X} + \Omega^2(t)X = 0. \tag{13}$$

The wave packet dynamics are now completely determined by the solutions to equations (12) and (13), which describe the time evolution of the width and centre of the packet, respectively. Integration of equations (12) and (13) are subject to the general initial conditions

$$a(0) = a_0, \qquad \dot{a}(0) = d_0 \tag{14}$$

$$X(0) = X_0, \qquad \dot{X}(0) = V_0. \tag{15}$$

Next, we show that the general solution to the quantum equation (12) can be obtained with the help of just one particular solution to the classical equation (13). To this end, let us now substitute

$$a(t) = r(\theta)\alpha(t) \tag{16}$$

and

$$\mathrm{d}\theta = \mathrm{d}t/\alpha^2(t) \tag{17}$$

into equation (16) to obtain

$$I_1 = [r'(\theta)]^2 + \frac{\hbar^2}{4m^2}\left(\frac{1}{r(\theta)}\right)^2 \tag{18}$$

provided that

$$\ddot{\alpha} + \Omega^2(t)\alpha = 0. \tag{19}$$

Upon a second integration equation (18) can be recast with the help of equation (16) as

$$a^2(t) = \left(\frac{\hbar^2}{4m^2I_1} + I_1I_2^2\right)\alpha_1^2(t) + I_1\alpha_2^2(t) + 2I_1I_2\alpha_1(t)\alpha_2(t) \tag{20}$$

where α_1 and α_2 are two independent solutions to the classical equation (19); I_1 and I_2 represent two invariants of motion of the problem.

In addition, the two independent solutions to equation (19) can be obtained in general from just one particular solution to the same equation, namely

$$\alpha_1(t) \equiv \alpha(t) \tag{21}$$

and

$$\alpha_2(t) \equiv \alpha(t)\int^t \frac{\mathrm{d}t'}{\alpha^2(t')}. \tag{22}$$

If initial conditions are imposed as follows:

$$\alpha(0) = 1 \qquad \text{and} \qquad \dot{\alpha}(0) = 0 \tag{23}$$

we are led to

$$\alpha_1(0) = 1 \qquad \alpha_2(0) = 0 \qquad \dot{\alpha}_1(0) = 0 \qquad \text{and} \qquad \dot{\alpha}_2(0) = 1. \tag{24}$$

These initial conditions allow us to find the two invariants of motion:

$$I_1 = d_0^2 + (a_0^2/\tau^2) \tag{25}$$

and

$$I_2 = a_0d_0/[d_0^2 + (a_0^2/\tau^2)]. \tag{26}$$

Thus, the complete dynamics of the TDHO can be found with the help of equations (21), (22), (25) and (26): the generalized squeezed states for the TDHO wave packet is finally written as

$$a(t) = a_0\alpha(t)\left\{1 + \left(\frac{2\varsigma}{\tau}\right)\left[\int^t \frac{\mathrm{d}t'}{\alpha^2(t')}\right] + \left(\frac{(1+\varsigma^2)}{\tau^2}\right)\left[\int^t \frac{\mathrm{d}t'}{\alpha^2(t')}\right]^2\right\}^{1/2} \tag{27}$$

where $\tau = 2ma_0^2/\hbar$, $\varsigma = 2ma_0d_0/\hbar$, $a(0) = a_0$ and $\dot{a}(0) = d_0$.

In turn, the packet centre of gravity evolves according to

$$X(t) = \alpha(t)\left\{X_0 + V_0 \int^t \frac{dt'}{\alpha^2(t')}\right\}. \quad (28)$$

Now, the full wave packet can now be written in its final, general form:

$$\psi(x,t) = (2\pi a^2(t))^{-1/4} \exp\left[-\frac{[x - X(t)]^2}{4a^2(t)}\right]$$
$$\times \exp\left\{\frac{i}{\hbar}\left[\frac{m\dot{a}}{2a}[x - X(t)]^2 + m\dot{X}(t)[x - X(t)]\right]\right\}$$
$$\times \exp\left\{\frac{i}{\hbar}\int_0^t dt'\left(\frac{1}{2}m\dot{X}^2(t') - \frac{1}{2}m\Omega^2 X^2(t') - \frac{\hbar^2}{4ma^2(t')}\right)\right\}. \quad (29)$$

The wave packet surrounds the position of the classical particle and the centre of gravity of the packet follows the classical trajectory: its time evolution is completely determined by the quantum and classical solutions (27) and (28), respectively. At the moment of observation, the packet is moving with an initial velocity V_0 and spreading with an initial rate d_0. The associated variance of x (squared uncertainty) can be written as $[\Delta x(t)]^2 = a^2(t)$, which exhibits the generalized squeezed states for the TDHO. Above all, our method gives a general solution to the quantum TDHO from just one particular solution $\alpha(t)$ to the classical equation (19).

To check the validity of our solution, let us consider as a particular case the TDHO treated by Mostafazadeh [1], Ji *et al* [2], Lo [3] and Agarwal and Kumar [6], for which the restoring force is linearly swept, i.e.

$$\Omega^2(t) = \begin{cases} \Omega_0^2 & \text{for} \quad -\infty < t < 0 \\ \Omega_0^2(1 + \beta_0 t/T) & \text{for} \quad 0 \leqslant t \leqslant T \\ \Omega_0^2(1 + \beta_0) & \text{for} \quad T < t < \infty. \end{cases} \quad (30)$$

The oscillator is initially in the ground state. Then, for $0 \leqslant t \leqslant T$, the oscillator passes through the intermediate sweeping frequency region. Finally, for $T < t < \infty$, the oscillator reaches the last region. With the help of equations (21), (22) and (27), the variance of the position (squared uncertainty) $a^2(t)$ of the new generalized squeezed states for this case can be explicitly calculated as follows[1].

For $-\infty < t < 0$,

$$a^2(t) = a_0^2\left\{\cos^2\Omega_0 t + \left(\frac{2\varsigma}{\tau\Omega_0}\right)\cos\Omega_0 t \sin\Omega_0 t + \left(\frac{1+\varsigma^2}{\tau^2\Omega_0^2}\right)\sin^2\Omega_0 t\right\}. \quad (31a)$$

For $0 \leqslant t \leqslant T$,

$$a^2(t) = a_0^2\eta\left\{J_{1/3}^2(z) + \left(\frac{2\varsigma}{\tau\Omega_0^2(T/\beta_0)}\right)\tilde{J}_{1/3}(z)\tilde{Y}_{1/3}(z) + \left(\frac{1+\varsigma^2}{\tau^2\Omega_0^4(T/\beta_0)^2}\right)\tilde{Y}_{1/3}^2(z)\right\}. \quad (31b)$$

For $T < t < \infty$,

$$a^2(t) = a_0^2\left\{\begin{array}{l} \cos^2\Omega_0\sqrt{(1+\beta_0)}t + \left(\frac{2\varsigma}{\tau\Omega_0}\right) \\ \times \cos\Omega_0\sqrt{(1+\beta_0)}t \sin\Omega_0\sqrt{(1+\beta_0)}t \\ + \left(\frac{1+\varsigma^2}{\tau^2\Omega_0^2(1+\beta_0)}\right)\sin^2\Omega_0\sqrt{(1+\beta_0)}t \end{array}\right\}. \quad (31c)$$

where $\tilde{J}_{1/3}(z)$ and $\tilde{Y}_{1/3}(z)$ are renormalized Bessel functions of order 1/3 that take into account the initial conditions (24), $z = (2/3)\{\beta_0(\Omega_0)^{1/2}/T\}\{t+(T/\beta_0)\}^{3/2}$, $\eta = (\Omega_0)^{1/2}(t+T/\beta_0)^{1/2}$ and τ and ζ are defined as in equation (27). If we let $\varsigma = 0$, the results of equations (31*a*)–(31*c*) reduce to those found by Mostafazadeh [1], Ji *et al* [2], Lo [3] and Agarwal and Kumar [6]. These new dispersive properties of the TDHO are not apparent within the approaches using the Heisenberg and Schrödinger pictures [1–6]. Finally, let us now consider the general coherent solution given by Malkin *et al* [7]. For simplicity, when their solution is reduced to the case of a simple harmonic oscillator[2] (with constant frequency), it yields a zero value for $\dot{a}(0)$ (i.e. $\varsigma = 0$). We are led to conclude that their general coherent solution (for an arbitrary time-dependent frequency) is therefore a special case of our equation (27).

To sum up, new generalized squeezed states for the TDHO are found through the theory of invariants. Our method gives a comparatively clearer picture than methods using evolution operators because we can establish a direct connection between the classical and quantum solutions. The additional significance of our method is that it is possible to find generalized quantum squeezed states from just one particular solution to the classical time-dependent oscillator.

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[1] The classical solutions can be readily found.
For $-\infty < t < 0$,

$$X(t) = X_0\cos\Omega_0 t + \frac{V_0}{\Omega_0}\sin\Omega_0 t.$$

For $0 \leqslant t \leqslant T$,

$$X(t) = X_0\sqrt{\eta}\tilde{J}_{1/3}(z) + \frac{V_0\sqrt{\eta}}{\Omega_0^2(T/\beta_0)}\tilde{Y}_{1/3}(z).$$

For $T < t < \infty$,

$$X(t) = X_0\cos\Omega_0\sqrt{(1+\beta_0)}t + \frac{V_0}{\Omega_0\sqrt{(1+\beta_0)}}\sin\Omega_0\sqrt{(1+\beta_0)}t.$$

[2] For the harmonic oscillator of mass m and constant frequency Ω_0 the width $a(t)$ (from equation (27)) is explicitly given by

$$a^2(t) = \frac{\hbar^2}{8m^2a_0^2\Omega_0^2}\left[(A+B) + \sqrt{(A-B)^2 + C^2}\cos[(2\Omega_0 t) - \theta]\right]$$

where $t = 2ma_0^2/\hbar$, $\varsigma = 2ma_0d_0/\hbar$, $a(0) = a_0$, $\dot{a}(0) = d_0$, $A = \Omega_0^2\tau^2$, $B = [1+\varsigma^2]$, $C = 2\Omega_0\tau\varsigma$ and $\theta = \arctan[C/(A-B)]$. If we let $\varsigma = 0$, the above equation reduces to the corresponding expression (in our notation) obtained in section 2 of the first publication in [7].