Schrödinger equation for an extended electron

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Abstract

A new quantum mechanical wave equation describing the dynamics of an extended electron is derived via Bohmian mechanics. The solution to this equation is found through a wave packet approach which establishes a direct correlation between a classical variable with a quantum variable describing the dynamics of the center of mass and the width of the electron wave packet. The approach presented in this paper gives a comparatively clearer picture than approaches using elaborative manipulation of infinite series of operators. It is shown that the new Schrödinger equation is free of any runaway solutions or any acausal responses.

PACS: 03.65.Ta, 03.65.Sq, 03.70.+k

About a century ago, Lorentz [1] and Abraham [2] argued that when an electron is accelerated, there are additional forces acting due to the electron's own electromagnetic field. However, the so-called Lorentz-Abraham equation for a point-charge electron

$$m\frac{dV}{dt} = \frac{2e^2}{3c^3}\frac{d^2V}{dt^2} + F_{ext} \qquad (1)$$

was found to be unsatisfactory because, for $F_{ext} = 0$, it admits runaway solutions. These solutions clearly violate the law of inertia.

Since the seminal works of Lorentz and Abraham, inumerous papers and textbooks have given great consideration to the proper equation of motion of an electron.[3]-[11] The problematic runaway solutions were circumvented by Sommerfeld [5] and Page [6] by going to an extended model. In the nonrelativistic case of a sphere with uniform surface charge, such an electron obeys in good approximation the difference-differential equation:[7]-[9]

$$m\frac{dV}{dt} = \frac{e^2}{3L^2c}\left[V(t-2L/c) - V(t)\right] + F_{ext}. \qquad (2)$$

This extended model is finite and causal if the electron size L is larger than the classical electron radius $r_e = e^2/mc^2$. I shall limit the discussion here to the sphere with uniform surface charge; the case of a volume charge is considerably more complicated and adds nothing to the understanding of the problem.

The dynamics of charges is a key example of the importance of obeying the validity limits of a physical theory. If classical equations can no longer be trusted at distances of the order of (or below) the Compton wavelength, what is the Schrödinger equation that can replace Equation (2)? Within QED, workers have not been able to derive an equation of motion and it is unclear whether QED can actually produce an equation of motion at all. This work proposes an answer to this problem in the nonrelativistic regime.

A new quantum mechanical wave equation describing the dynamics of an extended electron is derived via Bohmian mechanics. The solution to this equation is found through a wave packet approach which establishes a direct correlation between a classical variable with a quantum variable describing the dynamics of the center of mass and the width of the electron wave packet. It is shown that the new Schrödinger equation is free of any runaway solutions or any acausal responses. Besides, the approach presented in this work gives a comparatively clearer picture than the modern time quantum approach carried out by Moniz and Sharp.[9] They derived an infinite order differential equation, i.e., an infinite series of derivatives that apparently cannot be summed.

The Bohmian interpretation of quantum mechanics[12]-[27] provides a framework for analyzing quantum systems by assuming that the wave function which satisfies Schrödinger's equation is no longer the most complete description of the state of the system. It ascribes a particle motion via the de Broglie guidance condition

$$\frac{dx}{dt} = v(x,t)\big|_{x=x(t)} = \frac{1}{m}\frac{\partial S}{\partial x}\bigg|_{x=x(t)} \qquad (3)$$

where v represents the particle velocity and S is the phase of the wave function ψ. By expressing the wave function in polar form as

$$\psi(x,t) = \phi(x,t)\exp(iS(x,t)/\hbar), \tag{4}$$

Schrödinger's equation can be recast as

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{m}\frac{\partial}{\partial x}\left(V_{ext} + V_{qu}\right), \tag{5}$$

and

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho v\right) = 0. \tag{6}$$

.

Equation (5) can be regarded as a modified Hamilton-Jacobi equation while Equation (6) is a continuity equation for $\rho = \phi^2$; V_{ext} denotes the external classical potential and

$$V_{qu} = -\frac{\hbar^2}{2m\phi}\frac{\partial^2 \phi}{\partial x^2} \tag{7}$$

is the so-called quantum potential.

Within the framework of Bohmian mechanics, a quantum extension to the Sommerfeld-Page equation (2) for an electron sphere with uniform surface charge in the absence of external forces can be accomplished by writing:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{e^2}{3mL^2c}\left[v(t-2L/c) - v(t)\right] + \frac{1}{m}\frac{\partial}{\partial x}\left(\frac{\hbar^2}{2m\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}\right). \tag{8}$$

Then, Equations (4) and (8) yield:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + \left\{\frac{i\hbar e^2}{6mL^2c}\ln\left(\frac{\psi(x,t-2L/c)\psi^*(x,t)}{\psi^*(x,t-2L/c)\psi(x,t)}\right)\right\}\psi(x,t). \tag{9}$$

In order to find the most general Gaussian wave packet solution to Equation (9), the following ansatz is made:

$$\rho(x,t) = \left(2\pi a^2(t)\right)^{-1/2}\exp\left[-\frac{[x-X(t)]^2}{2a^2(t)}\right], \tag{10}$$

where $a(t)$ and $X(t)$ are auxiliary functions of time, to be determined in what follows: they represent the width and center of mass of wave packet, respectively.

First, Equation (10) is substituted into (6) and integrated the result to obtain

$$v(x,t) = \frac{\dot{a}(t)}{a(t)}[x - X(t)] + \dot{X}(t) \tag{11}$$

where the constant of integration must be zero since ρ and $\rho\,(\partial S/\partial x)$ vanish for $|x| \to \infty$. In fact, any well-behaved function of $(x - X)$ multiplied by ρ clearly vanishes as $|x| \to \infty$. Then, substitution of Equations (10) and (11) into Equation (8) yields

$$\frac{m}{2a(t)}\left\{\ddot{a}(t) - \frac{e^2}{3mL^2c}\left[\dot{a}(t-2L/c) - \dot{a}(t)\right] - \frac{\hbar^2}{4m^2a^3(t)}\right\}[x - X(t)]^2 + \\ m\left\{\ddot{X}(t) - \frac{e^2}{3mL^2c}\left[\dot{X}(t-2L/c) - \dot{X}(t)\right]\right\}[x - X(t)] = 0. \tag{12}$$

This polynomial equation is satisfied once the coefficients of $[x - X(t)]$ and $[x - X(t)]^2$ are set equal to zero, namely,

$$\ddot{X}(t) - \frac{e^2}{3mL^2c}\left[\dot{X}(t-2L/c) - \dot{X}(t)\right] = 0 \tag{13}$$

and

$$\ddot{a}(t) - \frac{e^2}{3mL^2c}\left[\dot{a}(t-2L/c) - \dot{a}(t)\right] = \frac{\hbar^2}{4m^2a^3(t)}. \tag{14}$$

The wave packet dynamics is now completely determined by the Equations (13) and (14). The first equation is the Sommerfeld-Page equation (describing here the time evolution of center of the wave packet) which does not have runaway solutions.[9] The second equation is a *new result* (describing here the time evolution of the width of the wave packet). This equation is free of any runaway solutions or any acausal response due of the restrictive term on the right hand side: physically this means that for $t > 2e^2/3mc^3$ (which is the time required for light to traverse the extended electron) this term settles down the dynamics of the wave packet.

To sum up, a new formalism to describe the nonrelativistic quantum dynamics of an extended electron has been presented in this work. The equations derived give a comparatively clearer picture than other formalisms using elaborative manipulation of infinite series of operators. The approach here is also reasonable because the electron is smeared out due to the uncertainty principle and has the appropriate feature of a wave packet. In fact, there exists as yet no proper formulation in quantum electrodynamics (and may not be possible within QED as a perturbation theory). The only calculation by QED was reported by Low,[28] who has not been able to derive an equation of motion and it is unclear whether QED can actually produce an equation of motion at all. To my knowledge, this work presents the only nonrelativistic Schrödinger equation available to describe the problem. Further, Equation (9) may be used to investigate quantum tunneling through potential barriers in light of the work developed by Denef et al. on the so-called classical tunneling.[29] Work in this direction is in progress.

Acknowledgments

It is a pleasure to thank K. Carde, J. Feulner, S. Layne, J. Patterson J. Reiner and N. Sofroniew for helpful discussions.

References

[1] H. A. Lorentz, La theórie eléctromagnetique de Maxwell et son application aux corps mouvemants *Arch. Neérl. Sci. Exactes Nat.* 25, 363552 (1892); *The Theory of Electrons* Leipzig: Teubner (1909) (2nd edition, 1916).

[2] M. Abraham, *Theorie der Elektrizitat* Vol II: Elektromagnetische Theorie der Strahlung, Leipzig, Teubner (1905).

[3] F. Rohrlich, *Am. J. Phys.* 65, 1051 (1997); 68, 1109 (2000). *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965). See Chapter 6.

[4] D. Griffiths, *Introduction to Electrodynamics* (Prentice Hall, 3rd Edition, N. Y., 2004). See Chapter 11.

[5] A. Sommerfeld, *Simplified deduction of the field and the forces of an electron moving in any given way* Akad. van Wetensch. te Amsterdam 13 (1904) (English translation 7, 346 (1905)).

[6] L. Page, *Phys. Rev.* 11, 376 (1918).

[7] K.-J. Kim and A. M. Sessler, *8th Workshop on Advanced Acceleration Concepts* July 5-11, 1998, Baltimore, MD.

[8] J. D. Jackson, *Classical Electrodynamics* J. Wiley, New York, 1998, 3rd edition. See Chapter 16.

[9] E.J. Moniz and D. H. Sharp, *Phys. Rev.* D 15, 2850 (1977).

[10] H. Levine, E. J. Moniz, and D. H. Sharp, *Am. J. Phys.* 45, 75 (1977).

[11] R. J. Cook *Am. J. Phys.* 52, 894 (1984).

[12] R. Tumulka, *Am. J. Phys.* 72, 1220 (2004). See references therein.

[13] Oleg V. Prezhdo and Craig Brooksby, *Phys. Rev. Lett.* 86, 3215 (2001).

[14] Y. Nogamia, F. M. Toyamab and W. van Dijk, *Phys. Lett.* A270, 279 (2000).

[15] A. R. Plastino, M. Casas and A. Plastino *Phys. Lett.* A281, 297(2001).

[16] N. C. Dias and J. N. Prata *Phys. Lett.* A302, 261 (2002).

[17] A. J. Makowski, , P. Pepowski and S. T. Dembiski *Phys. Lett.* A266, 241 (2000).

[18] L. Shifren, , R. Akis and D. K. Ferry *Phys. Lett.* A274, 75 (2000).

[19] P. Falsaperla and G. Fonte *Phys. Lett.* A316, 382 (2003).

[20] A. Datta, P. Ghose and M. K. Samal *Phys. Lett.* A322, 277 (2004).

[21] C. Colijn and E. R. Vrscay *Phys. Lett.* A300, 334 (2002).

[22] G. Potel, M. Muoz-Alear, F. Barranco and E. Vigezzi *Phys. Lett.* A299, 125 (2002).

[23] R. G. Stomphorst *Phys. Lett.* A292, 213 (2002).

[24] Md. M. Ali, A. S. Majumdar and D. Home *Phys. Lett.* A304, 61 (2002).

[25] G. E. Bowman *Phys. Lett.* A298, 7 (2002).

[26] D. Dürr, S. Goldstein, R. Tumulka, and N. Zanghi, *Phys. Rev. Lett.* 93, 090402 (2004). See references therein.

[27] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987).

[28] F. Low, *Run-Away Electrons in Relativistic Spin 1/2 Quantum Electrodynamics* Preprint MIT- CTP-2522 (1997).

[29] F. Denef, J. Raeymaekers, U. M. Studer and W. Troost, *Phys. Rev.* E56, 3624 (1997).